# HEAT CONDUCTION FOR OFF-CENTER ANNULUS

V. S. Kolesov, I. I. Fedik,
and E. E. Chuiko

UDC 536.24.02

The stationary heat-conduction problem for an isotropic heat emitting off-center annulus is solved by using conformal transformations; the problem was analyzed in [1] for the particular case of constant heat emission. A similar problem for an off-center annulus with straight-line anisotropy is also solved by using the small-parameter method.

1. Let an arbitrary temperature distribution be specified on the boundaries $\Gamma_1$, $\Gamma_2$ of a heat-emitting off-center annulus with eccentricity d. The heat-conduction problem is formulated as follows: to find a function T(x, y) which satisfies the equation

$$\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q(x, y)}{\lambda} \tag{1.1}$$

as well as the boundary conditions

$$T|_{\Gamma_1} = f_1(x, y), \ T|_{\Gamma_2} = f_2(x, y). \tag{1.2}$$

The solution of (1.1) and (1.2) is sought in the form of a sum T = U + V, where U and V satisfy the following:

$$\Delta U = 0, \ U|_{\Gamma_1} = f_1(x, y), \ U|_{\Gamma_2} = f_2(x, y), \tag{1.3}$$

$$\Delta V = -\frac{Q(x, y)}{\lambda}, \ V|_{\Gamma_1} = V|_{\Gamma_2} = 0. \tag{1.4}$$

Let us first consider the problem (1.3). It is obvious that without loss of generality it is sufficient to find a solution which satisfies on one circle only (for example, on $\Gamma_1$) the boundary condition, and vanishes on the other circle.

The off-center annulus is mapped into a concentric one by means of the function [2]

$$w = \frac{z + c}{z - c}, \tag{1.5}$$

where the real value c and thus the origin of the coordinate system are determined by the condition that the points c and –c are symmetric relative to both circles, that is,

$$b_1 = \frac{1}{2}\left(\frac{R_1^2 - R_2^2}{d} + d\right), \quad b_2 = \frac{1}{2}\left(\frac{R_1^2 - R_2^2}{d} - d\right),$$

$$c = \sqrt{b_1^2 - R_1^2} = \sqrt{b_2^2 - R_2^2}.$$

The circle $\Gamma_1$ is mapped into the circle of radius $\gamma_1 = (b_1 + c)/R_1$, and the circle $\Gamma_2$ into the circle $\gamma_2 = (b_2 + c)/R_2$ where $\gamma_2 > \gamma_1 > 1$. The Laplace equation is invariant with respect to conformal mappings therefore, in accordance with the remarks made above it is sufficient that the solution be found of the following problem:

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} = 0, \ U|_{\rho=\gamma_1} = f(\varphi), \ U|_{\rho=\gamma_2} = 0, \tag{1.6}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 27, No. 6, pp. 1122-1127, December, 1974. Original article submitted May 25, 1974.

where

$$\rho = \left|\frac{z+c}{z-c}\right| = \sqrt{\frac{(x+c)^2+y^2}{(x-c)^2+y^2}},$$

$$\varphi = \operatorname{arctg}\left[\operatorname{Im}\left(\frac{z+c}{z-c}\right) \Big/ \operatorname{Re}\left(\frac{z+c}{z-c}\right)\right].$$

The solution of the problem (1.6) can easily be found by using the method of the separation of variables [3]:

$$U = \frac{a_0}{2\ln\frac{\gamma_1}{\gamma_2}}\ln\frac{\rho}{\gamma_2} + \sum_{n=1}^{\infty}\frac{\gamma_1^n(\gamma_2^{2n}-\rho^{2n})}{\rho^n(\gamma_2^{2n}-\gamma_1^{2n})}(a_n\cos n\varphi + b_n\sin n\varphi), \tag{1.7}$$

$a_0$, $a_n$, $b_n$ being the coefficients of the Fourier expansion for the function $f(\varphi)$.

In the case of isothermic surfaces ($\Gamma_1 \sim T_1$, $\Gamma_2 \sim T_2$) the temperature distribution is given by

$$H = \frac{T-T_1}{T_2-T_1} = \frac{\ln\rho - \ln\gamma_1}{\ln\gamma_2 - \ln\gamma_1}. \tag{1.8}$$

By setting H = $\alpha$ ($0 < \alpha < 1$) the isotherms are found, namely,

$$\left[x - c\coth\left(\alpha\ln\frac{\gamma_2^{\frac{\alpha-1}{\alpha}}}{\gamma_1}\right)\right]^2 + y^2 = \frac{c^2}{\operatorname{sh}^2\left(\alpha\ln\frac{\gamma_2^{\frac{\alpha-1}{\alpha}}}{\gamma_1}\right)}.$$

The problem (1.4) is now considered. Its solution can be reduced to the solution of the problem (1.3) provided a particular solution of the Poisson equation can be found for a given heat emitting function. In the general case, however, if the Poisson equation is transformed by (1.5) one obtains

$$\Delta V = -\frac{Q(\rho,\varphi)}{|W_z'|^2\lambda}, \tag{1.9}$$

where

$$|W_z'|^2 = \frac{(1-2\rho\cos\varphi+\rho^2)^2}{4c^2}.$$

By expanding the right-hand side of (1.9) into a Fourier series one seeks a solution by means of the separation of variables. By satisfying the boundary conditions (1.4), one finally obtains the following expression for the temperature:

$$V = \frac{1}{2\lambda}\left\{\frac{\ln\frac{\rho}{\gamma_1}}{\ln\frac{\gamma_2}{\gamma_1}}\int_{\gamma_1}^{\gamma_2}\eta a_0(\eta)\ln\frac{\rho}{\eta}\,d\eta - \int_{\gamma_1}^{\rho}\eta a_0(\eta)\ln\frac{\rho}{\eta}\,d\eta + \sum_{n=1}^{\infty}\left[\frac{\gamma_2^n}{\rho^n}\frac{\rho^{2n}-\gamma_1^{2n}}{\gamma_2^{2n}-\gamma_1^{2n}}\int_{\gamma_1}^{\gamma_2}\eta(\rho^n\eta^{-n}-\rho^{-n}\eta^n)\right.\right.$$

$$\left.\left.\times\frac{a_n(\eta)\cos n\varphi + b_n(\eta)\sin n\varphi}{n}\,d\eta - \int_{\gamma_1}^{\rho}\eta(\rho^n\eta^{-n}-\rho^{-n}\eta^n)\frac{a_n(\eta)\cos n\varphi + b_n(\eta)\sin n\varphi}{n}\,d\eta\right]\right\}, \tag{1.10}$$

$a_0$, $a_n$, $b_n$ being the Fourier coefficients for the right-hand side of Eq. (1.9).

For the case of constant heat emission, which is important in application, the solution of the problem (1.4) is given by

$$V = -\frac{Q}{2\lambda}\left\{\frac{1}{2}[(x-b)^2+y^2-R_1^2] + \frac{cd\ln\frac{\rho}{\gamma_1}}{\ln\frac{\gamma_2}{\gamma_1}} + 2cd\sum_{n=1}^{\infty}\frac{\rho^{2n}-\gamma_1^{2n}}{\gamma_2^{2n}-\gamma_1^{2n}}\frac{\cos n\varphi}{\rho^n}\right\}. \tag{1.11}$$

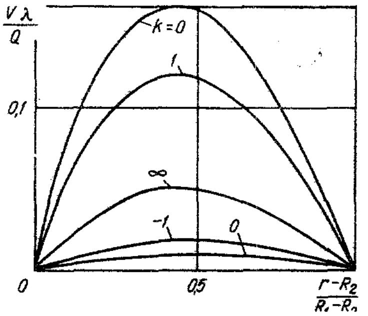


Fig. 1

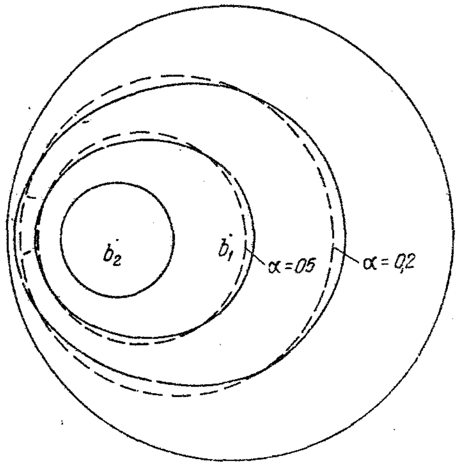


Fig. 2

Fig. 1. Temperature distribution in an off-center annulus.

Fig. 2. Isotherms in an off-center anisotropic annulus (continuous lines) and in an isotropic annulus (dashed lines).

Employing the Cauchy – Hadamard theorem it can easily be shown that the above series is absolutely convergent in the annulus $\gamma_1^2/\gamma_2^2 < \rho < \gamma_2^2/\gamma_1$.

In Fig. 1 graphs are shown of the temperature distribution in off-center annulus with constant heat emission along the rays from the center of the circle $\Gamma_2$. The label k indicates the tangent of the angle between the ray and the abscissa axis $Ox$, read counterclockwise. The calculations have been carried out for the following dimensions: $R_1 = 1$, $R_2 = 0.25$, $d = 0.5$.

2. The obtained solutions can also be employed to determine the temperature in an off-center annulus with a straight-line anisotropy. For example, the following problem is considered:

$$\lambda_1 \frac{\partial^2 T}{\partial x^2} + \lambda_2 \frac{\partial^2 T}{\partial y^2} = 0, \quad T|_{\gamma_1} = f(x, y), \quad T|_{\gamma_2} = 0. \tag{2.1}$$

In the above $\lambda_1$, $\lambda_2$ are the heat-conduction coefficients along the $Ox$, $Oy$ axes, respectively.

It is assumed without loss of generality that $\lambda_1 > \lambda_2$, and, consequently, that $\lambda_2/\lambda_1 < 1$, that is, it can be written as $\lambda_2/\lambda_1 = 1 - \varepsilon$ where $0 < \varepsilon < 1$. Equation (2.1) can now be rewritten as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \varepsilon \frac{\partial^2 T}{\partial y^2}. \tag{2.2}$$

The solution of (2.2) is sought in the form of power series in the powers of $\varepsilon$:

$$T = \sum_{n=0}^{\infty} \varepsilon^n a_n(x, y).$$

To determine $a_n(x, y)$ the problems are obtained,

$$\Delta a_0 = 0, \quad a_0|_{\Gamma_1} = f(x, y), \quad a_0|_{\Gamma_2} = 0, \tag{2.3}$$

$$\Delta a_n = \frac{\partial^2 a_{n-1}}{\partial y^2}, \quad a_n|_{\Gamma_1} = 0, \quad a_n|_{\Gamma_2} = 0 \quad (n \geqslant 1) \tag{2.4}$$

or, by introducing the complex variables $z = x + iy$, $\bar{z} = x - iy$,

$$4 \frac{\partial^2 a_n}{\partial z \partial \bar{z}} = \left( -\frac{\partial^2}{\partial z^2} + 2 \frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\partial^2}{\partial \bar{z}^2} \right) a_{n-1}. \tag{2.5}$$

The zeroth approximation $a_0(x, y)$ is given by the formula (1.7). By substituting the latter into (2.5), one obtains an inhomogeneous equation for determining the first approximation $a_1(x, y)$; its solution is represented in the following manner:

$$a_1(x, y) = \psi(z, \bar{z}) + u(x, y),$$

where $\psi(z, \bar{z})$ is the particular solution determined by the right-hand side of (2.5), and $U(x, y)$ is a harmonic function whose values on the circumference of the off-center annulus are adopted in such a way that the boundary conditions of the problem (2.4) are satisfied. Subsequent successive approximations can be obtained similarly. If the finding of a particular solution of (2.5) is very difficult one can employ the formula (10) to find the first and subsequent approximations.

To give an example, one finds the first approximation for the problem (2.1) by setting the temperature on the curve $\Gamma_1$ equal to the constant value $T_0$.

The zeroth approximation $a_0(x, y)$ is obtained from the formula (1.8); the approximation can be written as follows:

$$a_0(x, y) = \frac{\beta}{2}\left[\ln\frac{z+c}{z-c} + \ln\frac{\bar{z}+c}{\bar{z}-c} - 2\ln\gamma_2\right], \quad \beta = \frac{T_0}{\ln\frac{\gamma_2}{\gamma_1}}.$$

By now using $a_0(x, y)$ one obtains for the first approximation the equation

$$\frac{\partial^2 a_1}{\partial z \partial \bar{z}} = -\frac{\beta c}{2}\left[\frac{z}{(z^2-c^2)^2} + \frac{\bar{z}}{(\bar{z}^2-c^2)^2}\right].$$

Hence,

$$a_1(x, y) = \frac{\beta c}{4}\left[\frac{z}{z^2-c^2} + \frac{\bar{z}}{\bar{z}^2-c^2}\right] + u(x, y).$$

By changing to the coordinates $\rho$, $\varphi$ and by expanding the expression in the square brackets into a Fourier series one finds $U(\rho, \varphi)$. Finally, the first approximation is given by

$$a_1(\rho, \varphi) = \frac{\beta}{4}\left\{-\frac{2(\rho^2-1)}{1-2\rho\cos\varphi+\rho^2} + \frac{1}{\ln\frac{\gamma_2}{\gamma_1}}\left(\frac{\gamma_2^2-1}{\gamma_2^2}\ln\frac{\rho}{\gamma_1} - \frac{\gamma_1^2-1}{\gamma_1^2}\ln\frac{\rho}{\gamma_2}\right) + \frac{(\gamma_1^2-1)^2}{\gamma_1^2}\sum_{k=1}^{\infty}\frac{\rho^{2k}-\gamma_1^{2k}}{\gamma_2^{2k}-\gamma_1^{2k}}\frac{\cos k\varphi}{\rho^k} - \frac{(\gamma_2^2-1)^2}{\gamma_2^2}\sum_{k=1}^{\infty}\frac{\rho^{2k-}-\gamma_1^{2k}}{\gamma_2^{2k}-\gamma_1^{2k}}\frac{\cos k\varphi}{\rho^k}\right\}. \quad (2.6)$$

The continuous lines in Fig. 2 show the isotherms in the anisotropic off-center annulus ($\varepsilon = 0.5$) using the first approximation; the dashed lines show the isotherms in the isotropic annulus. The anisotropy causes a deformation of the circular isotherms for an isotropic annulus by extending or compressing them, respectively, in the direction of greater or smaller conductivity. In the example under consideration the temperature difference reaches 15%.

Here it should be mentioned that in computing the series (1.11) and (2.6) it is sufficient that the first few terms be calculated corresponding to those indices k for which the quantity $(\gamma_2/\gamma_1)^{2k} \gg 1$; the general term can be represented as $\mathrm{Re}\,(\mu e^{i\varphi})^k$ and the sum of the remainder of the series can therefore be found as a sum of a geometrical progression.

## NOTATION

| | |
|---|---|
| $x, y$ | are the Cartesian coordinates; |
| $b_1, b_2$ | are the distances between the centers of the circles and the origin; |
| $R_1, R_2$ | are the radii of the circles; |
| $T$ | is the temperature; |
| $Q$ | is the three-dimensional heat emission; |
| $\lambda$ | is the thermal conductivity; |
| $z = x + iy$ | is the point of the complex plane z; |
| $i = \sqrt{-1}$ | is the imaginary unit; |
| $\mathrm{Im}\, f(z)$ | is the imaginary part of complex function f(z); |
| $\mathrm{Re}\, f(z)$ | is the real part of complex function f(z). |

## LITERATURE CITED


1. I. S. Akimov, Inzh.-Fiz. Zh., No. 2, 227 (1965).
2. M. A. Lavrent'ev and V. B. Shabat, Methods of the Theory of Functions of the Complex Variable [in Russian], Nauka, Moscow (1965).
3. L. V. Kantorovich and V. I. Krylov, Approximation Methods of Higher Analysis [in Russian], Fizmatgiz, Moscow (1962).
4. A. I. Markushevich, Theory of Analytic Functions [in Russian], Moscow – Leningrad (1950).
5. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series and Products [in Russian], Fizmatgiz, Moscow (1969).